Ranking of Exponential Vague Sets With an Application to Decision Making Problems

Pushpinder Singh
Department of Computer Science,
Palacky University, 17. listopadu 12, CZ-77146,
Olomouc, Czech Republic.
pushpindersnl@gmail.com

ABSTRACT
The main aim of this paper is to propose a new approach for the ranking of exponential vague sets. The concepts of exponential vague sets and arithmetic operations between two exponential vague sets are introduced. The shortcomings of some existing ranking approaches for the ranking of generalized fuzzy sets and intuitionistic fuzzy numbers are pointed out. The proposed method considers not only the rank but also the decision maker’s optimistic attitude and it is shown that proposed ranking approach is more intuitive and reasonable as compared to existing ranking approaches. Also the proposed ranking function satisfies the reasonable properties for the ordering of fuzzy quantity. For practical use, proposed ranking approach is applied to decision making problem.

Keywords: Fuzzy sets, vague sets, exponential vague sets, intuitionistic fuzzy numbers, ranking functions, decision making problems.

RESUMEN
El objetivo principal de este trabajo es proponer un nuevo enfoque para la clasificación de los conjuntos inciertos exponenciales. Se introducen los conceptos de conjuntos inciertos exponenciales y operaciones aritméticas entre dos conjuntos inciertos exponenciales. Se señalan las deficiencias de algunos enfoques de clasificación existentes para la clasificación de los conjuntos difusos generalizados y de los números difusos intuicionistas. El método propuesto toma en cuenta no sólo el rango, sino también el enfoque optimista para toma de decisiones y se muestra que el enfoque de clasificación propuesto es más intuitivo y razonable en comparación con los enfoques de clasificación existentes. Asimismo, la función de clasificación propuesta satisface las propiedades razonables para el ordenamiento de la cantidad difusa. Para usos prácticos, el enfoque de clasificación propuesto se aplica al problema de toma de decisiones.

1. Introduction

The theory of fuzzy sets was first introduced by Zadeh [37] in 1965. Since then, the theory of fuzzy sets is applied in many fields such as pattern recognition, control theory, management sciences and picture processing, etc. In the field of fuzzy mathematics many mathematical theory such as fuzzy optimization, fuzzy topology, fuzzy logic, fuzzy analysis and fuzzy algebra etc. are obtained [3, 10, 14, 22, 23, 29, 32, 35]. In many applications of fuzzy set theory to decision making, we are faced with the problem of selecting one from a collection of possible solutions and in general we want to know which one is the best. This selection process may require that we rank or order fuzzy numbers. In order to rank fuzzy numbers, one fuzzy number needs to be evaluated and compared to others but this may not be easy. As known, the real numbers in can be linearly ordered by, however, fuzzy numbers cannot be done in such a way. Since fuzzy numbers are represented by possibility distributions, they can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than the other.

To the task of comparing fuzzy numbers, many authors proposed fuzzy ranking methods [5, 6, 8, 9, 12, 15, 16, 17, 18, 20, 25, 30, 31, 34]. But among all the methods, most of them consider only one point of view on comparing fuzzy quantities in spite of the different demand of the decision maker, so some improved methods have been brought forward which lead to produce different rankings for the same problem. Until now, there have not one unify method to this problem. Fuzzy set theory [37] has been shown to be useful tool to
handle the situations, in which the data is imprecise, by attributing a degree to which a certain object belongs to a set. In real life, a person may assume that an object belongs to a set, but it is possible that he is not sure about it. In other words, there may be hesitation or uncertainty that whether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or uncertainty. A possible solution is to use intuitionistic fuzzy set [2] and vague set [11]. Bustince and Burillo [4] pointed out that the notion of vague set is the same as that of intuitionistic fuzzy set. Lu and Ng [21] proved that vague sets is more natural than using an intuitionistic fuzzy set. Several authors [19, 24, 27, 28] have proposed different methods for the ranking of intuitionistic fuzzy sets but to the best of our knowledge till now there no method in the literature for the ranking of vague sets.

The main aim of this paper is to propose a new approach for the ranking of exponential vague sets. The concepts of exponential vague sets and arithmetic operations between two exponential vague sets are introduced. The shortcomings of some existing ranking approaches [8, 19] for the ranking of generalized fuzzy sets and intuitionistic fuzzy sets, vague sets and arithmetic operations between vague sets are pointed out. Also it is shown that proposed ranking approach is more intuitive and reasonable as compared to existing ranking approaches Rest of the paper is organized as follows: In Section 2, some basic definitions related to generalized fuzzy sets, intuitionistic fuzzy sets, vague sets and arithmetic operations between vague sets are presented. In Section 3, a brief review of the existing approach [8] for the ranking of generalized trapezoidal fuzzy numbers and the existing approach [19] for the ranking of intuitionistic fuzzy numbers are presented. In Section 4, the shortcomings of existing approaches [8, 19] are discussed. In Section 5, a new approach is proposed for the ranking of exponential vague sets. In Section 6, it is proved that the proposed ranking function satisfies the reasonable properties for the ordering of fuzzy quantities and results are compared with some existing approached. In Section 7, an application of proposed ranking method to decision making is presented. Section 8 draws the conclusions.

2. Preliminaries

In this section some basic definitions related to generalized fuzzy sets, intuitionistic fuzzy sets, vague sets and arithmetic operations between exponential vague sets are presented.

2.1 Generalized Fuzzy Sets

Definition 1. [6] A fuzzy set $A$, defined on the universal set of real numbers $R$, is said to be a generalized fuzzy number if its membership function has the following characteristics:
1. $\mu_A : R \rightarrow [0, w]$ is continuous.
2. $\mu_A(x) = 0$, for all $x \in (-\infty, a] \cup [d, \infty)$.
3. $\mu_A(x)$ is strictly on $[a, b]$ and strictly decreasing on $[c, d]$.
4. $\mu_A(x) = w$, for all $x \in [b, c]$, where $0 < w \leq 1$.

Definition 2. [6] A generalized fuzzy number, denoted as $A = (a, b, c, d; w)$ is said to be a generalized trapezoidal fuzzy number if its membership function is given by

$$
\mu_A(x) = \begin{cases} 
\frac{w(x-a)}{(b-a)}, & a \leq x < b, \\
\frac{w}{(b-a)}, & b \leq x \leq c, \\
\frac{w(x-c)}{(b-d)}, & c < x \leq d, \\
0, & \text{otherwise}
\end{cases}
$$

2.2 Intuitionistic Fuzzy Sets

Definition 3. [19] An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ on the universal set $X$ is characterized by a truth membership function $\mu_A(x), \mu_A(x) : X \rightarrow [0,1]$ and a false membership $\nu_A(x), \nu_A(x) : X \rightarrow [0,1]$. The values $\mu_A(x)$ and $\nu_A(x)$ represents the degree of membership and degree of non-membership of $x$ and always satisfies the condition $\mu_A(x) + \nu_A(x) \leq 1$. The value $1 - \mu_A(x) + \nu_A(x)$ represents the degree of hesitation of $x \in X$. 

Definition 4. [19] An intuitionistic fuzzy set $A$, defined on the universal set of real numbers $R$, said to be a triangular intuitionistic fuzzy number, denoted as $A = (a, b, c; w, u)$, if degree of membership and degree of non-membership are given by:

$$
\mu_A(x) = \begin{cases} 
\frac{w(x-a)}{(b-a)}, & a \leq x < b, \\
w, & x = b, \\
\frac{w(c-x)}{(c-b)}, & b < x \leq c, \\
0, & \text{otherwise}
\end{cases}
$$

and

$$
\nu_A(x) = \begin{cases} 
\frac{(b-x+u(x-a))}{(b-a)}, & a \leq x < b, \\
u, & x = b, \\
\frac{(x-b+u(c-x))}{(c-b)}, & b < x \leq c, \\
1, & \text{otherwise}
\end{cases}
$$

respectively. The values $w$ and $u$ represent the supremum of the degree of membership and the infimum of the degree of non-membership, respectively.

2.3 Vague sets and exponential vague sets

Definition 5. [11] A vague set $A = \{(x, \mu_A(x), 1-\nu_A(x)) \mid x \in X\}$ on the universal set $X$ is characterized by a truth membership function $\mu_A(x), \mu_A(x) : X \to [0,1]$ and a false membership $\nu_A(x), \nu_A(x) : X \to [0,1]$. The values $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and degree of non-membership of $x$ and always satisfies the condition $\mu_A(x) + \nu_A(x) \leq 1$. The value $1 - \mu_A(x) - \nu_A(x)$ represents the degree of hesitation of $x \in X$.

Definition 6. [11] A vague set $A$, defined on the universal set of real numbers $R$, denoted as $A = (a, b, c, d; \lambda, \rho)$, where $a \leq b \leq c \leq d$ and $\lambda \leq \rho$, is said to be a triangular vague set if degree of membership, $\mu_A(x)$, and complement of the degree of non-membership, $1-\nu_A(x)$, are given by

$$
\mu_A(x) = \begin{cases} 
\frac{\lambda(x-a)}{(b-a)}, & a \leq x < b, \\
\lambda, & x = b, \\
\frac{\lambda(x-c)}{(b-c)}, & b < x \leq c, \\
0, & \text{otherwise}
\end{cases}
$$

and

$$
1 - \nu_A(x) = \begin{cases} 
\frac{\rho(x-a)}{(b-a)}, & a \leq x < b, \\
\rho, & x = b, \\
\frac{\rho(x-c)}{(b-c)}, & b < x \leq c, \\
0, & \text{otherwise}
\end{cases}
$$

Definition 7. A vague set $A$, defined on the universal set of real numbers $R$, denoted as $A = (a, b, c, d; \lambda, \rho)$, where $a \leq b \leq c \leq d$ and $\lambda \leq \rho$, is said to be an exponential vague set if degree of membership, $\mu_A(x)$, and complement of the degree of non-membership, $1-\nu_A(x)$, are given by

$$
\mu_A(x) = \lambda \exp\left(-\frac{(b-x)}{(b-a)}\right), & a \leq x < b, \\
\lambda, & b \leq x \leq c, \\
\lambda \exp\left(-\frac{(x-c)}{(d-c)}\right), & c < x \leq d, \\
0, & \text{otherwise}
$$

and

$$
1 - \nu_A(x) = \rho \exp\left(-\frac{(b-x)}{(b-a)}\right), & a \leq x < b, \\
\rho, & b \leq x \leq c, \\
\rho \exp\left(-\frac{(x-c)}{(d-c)}\right), & c < x \leq d, \\
0, & \text{otherwise}
$$
2.4 Arithmetic operations between exponential vague sets

Let $A = (a_1,b_1,c_1,d_1;\lambda_1,\mu_1)$ and $B = (a_2,b_2,c_2,d_2;\lambda_2,\mu_2)$ be two exponential vague sets then

(i) $A \ominus B = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \min(\lambda_1, \lambda_2), \min(\mu_1, \mu_2))$

(ii) $A \Theta B = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; \min(\lambda_1, \lambda_2), \min(\mu_1, \mu_2))$

(iii) $aA = \begin{cases} (a_{11},a_{12},a_{13},a_{14},\lambda_1,\mu_1), & \alpha \geq 0 \\ (a_{21},a_{22},a_{23},a_{24},\lambda_2,\mu_2), & \alpha < 0 \end{cases}$

3. Overview of existing ranking approaches

In this section, the existing ranking approach [8, 19] are briefly discussed.

3.1 Chen and Sanguansat ranking approach

Let $A = (a_1,b_1,c_1,d_1;w_1)$ and $B = (a_2,b_2,c_2,d_2;w_2)$ be two generalized fuzzy numbers, then use the following steps to compare $A$ and $B$:

**Step 1.** Standardize each generalized fuzzy number $A$ and $B$ into $A'$ and $B'$ as follows:

$$A' = \left(\frac{a_1}{k}, \frac{b_1}{k}, \frac{c_1}{k}, \frac{d_1}{k}; w'_A\right) = (a'^*_1, b'^*_1, c'^*_1, d'^*_1; w'_A)$$ (1)

Where, $k = \max\left(\max\{a_1, b_1, c_1, d_1\}, \alpha_1\right)$, $\alpha_1$ denoted the absolute value of $a_1, b_1, c_1, d_1$ respectively, and $\min\{a_1, b_1, c_1, d_1\}$ denoted the upper bound of $[a_1, b_1, c_1, d_1]$ respectively.

$$B' = \left(\frac{a_2}{k}, \frac{b_2}{k}, \frac{c_2}{k}, \frac{d_2}{k}; w'_B\right) = (a'^*_2, b'^*_2, c'^*_2, d'^*_2; w'_B)$$ (2)

Then, calculate the areas $AreaA'_1$ and $AreaA'_2$, respectively, which denote the areas from the membership function curves of $\mu_1^l$ and $\mu_1^u$ respectively, to the membership function curve of the generalized fuzzy number $(-1,-1,-1,-1;w_1)$, where

$$AreaA'_1 = w_1 \left(\frac{1 - a_1^*}{2} + \frac{1 - b_1^*}{2}\right)$$ (7)

$$AreaA'_2 = w_1 \left(\frac{1 - c_2^*}{2} + \frac{1 - d_2^*}{2}\right)$$ (8)

Similarly, for generalized fuzzy set $B$

$$AreaB'_1 = w_2 \left(\frac{1 - a_2^*}{2} + \frac{1 - b_2^*}{2}\right)$$ (9)

$$AreaB'_2 = w_2 \left(\frac{1 - c_2^*}{2} + \frac{1 - d_2^*}{2}\right)$$ (10)
Step 3. Calculate the values
\[
XI_{A'} = \text{Area}_A^L + \text{Area}_A^U
\]
\[
XD_{A'} = \text{Area}_A^C + \text{Area}_A^U
\]
and
\[
XI_{B'} = \text{Area}_B^L + \text{Area}_B^U
\]
\[
XD_{B'} = \text{Area}_B^C + \text{Area}_B^U
\]

Step 4. Calculate the ranking score \(\text{Score}(A')\) and \(\text{Score}(B')\) of each generalized trapezoidal fuzzy number \(A'\) and \(B'\) as follows:
\[
\text{Score}(A') = \frac{XI_{A'} - XD_{A'}}{XI_{A'} + XD_{A'} + (1 - w_A)}
\]
and
\[
\text{Score}(B') = \frac{XI_{B'} - XD_{B'}}{XI_{B'} + XD_{B'} + (1 - w_B)}
\]

The following three cases may arise
(i) If \(\text{Score}(A') > \text{Score}(B')\), then \(A \succ B\).
(ii) If \(\text{Score}(A') < \text{Score}(B')\), then \(A \prec B\).
(iii) If \(\text{Score}(A') = \text{Score}(B')\), then \(A \sim B\).

3.2 Li ranking approach

Let \(A = (a_1, b_1, c_1; w_1, u_1)\) and \(B = (a_2, b_2, c_2; w_2, u_2)\) be two triangular intuitionistic fuzzy numbers then use the following steps to compare \(A\) and \(B\):

Step 1. Calculate
\[
V(A, \alpha) = V_\mu(A) + \alpha(V_\nu(A) - V_\mu(A)),
\]
\[
A(A, \alpha) = A_\mu(A) - \alpha(A_\nu(A) - A_\mu(A)),
\]
where
\[
V_\mu(A) = \frac{w_1(a_1 + 4b_1 + c_1)}{6},
\]
\[
V_\nu(A) = \frac{(1 - u_1)(a_1 + 4b_1 + c_1)}{6},
\]
\[
A_\mu(A) = \frac{(1 - u_1)(c_1 - a_1)}{3}, \quad \alpha \in [0, 1]
\]
and
\[
V(B, \alpha) = V_\mu(B) + \alpha(V_\nu(B) - V_\mu(B)),
\]
\[
A(B, \alpha) = A_\mu(B) - \alpha(A_\nu(B) - A_\mu(B)),
\]
where
\[
V_\mu(B) = \frac{w_2(a_2 + 4b_2 + c_2)}{6},
\]
\[
V_\nu(B) = \frac{(1 - u_2)(a_2 + 4b_2 + c_2)}{6},
\]
\[
A_\mu(B) = \frac{(1 - u_2)(c_2 - a_2)}{3}, \quad \alpha \in [0, 1]
\]

Step 2. Calculate
\[
\Re(A, \alpha) = \frac{V(A, \alpha)}{1 + A(A, \alpha)} \quad \text{and} \quad \Re(b, \alpha) = \frac{V(B, \alpha)}{1 + A(B, \alpha)}
\]
The following three cases may arise
(i) If \(\Re(A, \alpha) < \Re(B, \alpha)\), then \(A \prec B\).
(ii) If \(\Re(A, \alpha) > \Re(B, \alpha)\), then \(A \succ B\).
(iii) If \(\Re(A, \alpha) = \Re(B, \alpha)\), then \(A \sim B\).

4. Shortcomings of existing ranking approaches

In this section, the shortcomings of the existing ranking approaches \([8, 19]\) are pointed out.

4.1 Shortcomings of Chen and Sanguansat ranking approach

Here we pointed out the shortcomings of Chen and Sanguansat ranking approach \([8]\), on the basis of
reasonable properties of fuzzy quantities [34] and on the basis of height of fuzzy numbers.

4.1.1 On the basis of reasonable properties for fuzzy quantities.

With the help of some examples it shown that the ranking function, proposed by Chen and Sanguansat [8] does not satisfy the reasonable property, \( A > B \implies AOB > BOB \), for the ordering of fuzzy quantities i.e., according to Chen and Sanguansat approach [8] \( A > B \implies AOB > BOB \), which is a contradiction according to Wang and Lee [34].

Example 1. Let \( A = (0.1,0.3,0.3,0.5;0.8) \), and \( B = (0.1,0.3,0.3,0.5;1) \) be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8] \( B > A \) but \( BOA < AOA \) i.e., \( B > A \implies BOA > AOA \).

Example 2. Let \( A = (0.1,0.3,0.3,0.5;1) \), and \( B = (0.3,0.5,0.5,0.7;1) \) be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8] \( B > A \) but \( BOA < AOA \) i.e., \( B > A \implies BOA > AOA \).

Example 3. Let \( A = (-0.5,-0.3,-0.3,-0.1;1) \), and \( B = (0.1,0.3,0.3,0.5;1) \) be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8] \( B > A \) but \( BOA < AOA \) i.e., \( B > A \implies BOA > AOA \).

4.1.2 On the basis of height of fuzzy numbers

In some cases Chen and Sanguansat approach [8] indicates that the ranking of fuzzy numbers depends upon height of fuzzy numbers while in several cases it indicates that the ranking of fuzzy number does not depend upon the height of fuzzy numbers, which is not reasonable.

Let \( A = (a_1, a_2, a_3, a_4; w_1) \) and \( B = (a_1, a_2, a_3, a_4; w_2) \) be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8]

Case (i) If \( a_1 + a_2 + a_3 + a_4 \neq 0 \) then

\[
\begin{align*}
A < B & \text{ if } w_1 < w_2, \\
A > B & \text{ if } w_1 > w_2, \\
A \sim B & \text{ if } w_1 = w_2.
\end{align*}
\]

Case (ii) If \( a_1 + a_2 + a_3 + a_4 = 0 \) then \( A \sim B \) for all values of \( w_1 \) and \( w_2 \).

Example 4. Let \( A = (1,1,1; w_1) \) and \( B = (1,1,1; w_2) \) be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8]

\( A < B \) if \( w_1 < w_2 \), \( A > B \) if \( w_1 > w_2 \), and \( A \sim B \) if \( w_1 = w_2 \).

Example 5. Let \( A = (-0.4,0.2,-0.1,0.7; w_1) \) and \( B = (-0.4,0.2,-0.1,0.7; w_2) \), be two generalized trapezoidal fuzzy numbers then \( A \sim B \) for all values of \( w_1 \) and \( w_2 \).

According to Chen and Sanguansat approach [8], in first case ranking of fuzzy numbers depends upon height and in second case ranking does not depend upon the height which is not reasonable.

4.2 Shortcomings of Li ranking approach

Li [19] pointed out the shortcomings of all the existing approaches for the ranking of intuitionistic fuzzy numbers and proposed a new approach for the ranking of intuitionistic fuzzy numbers. In this section, the shortcomings of existing approach [19] are pointed out.

(i) The existing results, presented in Theorem 1 to Theorem 6 [19], are correct only if both \( \sim A \) and \( \sim B \) are intuitionistic fuzzy numbers with same degree of membership and the same degree of non-membership but in the real life problems, we need to compare the intuitionistic fuzzy numbers with different degree of membership and degree of non-membership.

If we have two intuitionistic fuzzy numbers with unequal degree of membership and degree of non-membership than the existing approach [19] can not be used, also in that case the ranking function does not satisfy the reasonable property,
Example 6. Let \( A = (1,3,5;0.3,0.2) \) and \( A = (4,8,9;0.4,0.1) \) be two triangular intuitionistic fuzzy numbers. According to existing ranking approach [19] the values of \( \mathfrak{R}(A,\frac{1}{2}) \) and \( \mathfrak{R}(B,\frac{1}{2}) \) are 0.6347 and 0.822 respectively so \( A < B \) but the values of \( \mathfrak{R}(A\Theta B) \) and \( \mathfrak{R}(B\Theta B,\alpha) \) are 0 respectively which is a contradiction.

Example 7. Let \( A = (2.0,3.0;5,3,1) \) and \( A = (1.0,4.0;9,8,4) \) be two triangular intuitionistic fuzzy numbers. According to existing ranking approach [19] the values of \( \mathfrak{R}(A,\frac{1}{2}) \) and \( \mathfrak{R}(B,\frac{1}{2}) \) are 0.6347 and 0.822 respectively so \( A < B \) but the values of \( \mathfrak{R}(A\Theta B) \) and \( \mathfrak{R}(B\Theta B,\alpha) \) are 0 respectively which is a contradiction.

Example 8. Let \( A = (-8,1,4;w_1,u_1) \) and \( B = (-8,1,4;w_2,u_2) \) be two intuitionistic fuzzy numbers then, according to existing approach [19] \( A \sim B \) for all values of \( w_1, u_1, w_2 \) and \( u_2 \).

Example 9. Let \( A = (1,1,1;w_1,u_1) \) and \( B = (1,1,1;w_2,u_2) \) be two intuitionistic fuzzy numbers then, according to existing approach [19] \( A \sim B \) for all values of \( w_1, u_1, w_2 \) and \( u_2 \).

Example 10. Let \( A = (a_1,b_1,c_1,d_1;\lambda_1,\rho_1) \) and \( B = (a_2,b_2,c_2,d_2;\lambda_2,\rho_2) \) be two exponential vague sets, where \( a_1 \leq b_1 \leq c_1 \leq d_1 \) and \( a_2 \leq b_2 \leq c_2 \leq d_2 \) then use the following steps to compare \( A \) and \( B \):

Step 1. Find \( (\lambda, \rho) = (\min(\lambda_1,\lambda_2), \min(\rho_1,\rho_2)) \).

Step 2. Find \( \mathfrak{R}(A,\lambda) = \int_0^\lambda (\alpha(b_1 + (b_1 - a_1) \log(\frac{x}{\lambda}))) + (1 - \alpha)(c_1 - (d_1 - c_1) \log(\frac{x}{\lambda})) \) \( \Rightarrow \mathfrak{R}(A,\lambda) = a\lambda + (1 - \alpha) d_1 \lambda \) and
\[ R(A_{p-\lambda}) = \int_{\lambda}^{\rho} \left\{ \alpha L_{p-\lambda}^{-1}(x) + (1-\alpha) R_{p-\lambda}^{-1}(x) \right\} dx, \text{ where} \]

\[ L_{p-\lambda}^{-1}(x) = b_1 - (h_1 - a_1) \log(x/\lambda) \log(\lambda/x), \]

\[ R_{p-\lambda}^{-1}(x) = c_1 + (d_1 - c_1) \log(x/\lambda) \log(\lambda/x), \]

\[ \Rightarrow R(A_{p-\lambda}) = \alpha \{ (h_1(\rho - \lambda) - (h_1 - a_1) \log(\lambda/\rho) \}
\]

\[ + (1 - \alpha) \{ (c_1(\rho - \lambda) - (d_1 - c_1) \log(\lambda/\rho) - (\rho - \lambda) \}, \text{ where } \alpha \in [0,1] \]

Corollary 1. If \( 0 \leq \lambda = \rho \leq 1 \), then (17) reduces to

\[ R(A) = \frac{\lambda(a_1 + d_1)}{2}, \text{ which is the ranking formula for generalized exponential fuzzy numbers [7].} \]

Corollary 2. If \( 0 \leq \lambda = \rho \leq 1 \) and left and right membership functions of \( A \) as linear functions, then (17) reduces to

\[ R(A) = \frac{1}{4} \left( \lambda(a_1 + d_1) + 2(h_1 - a_1 + c_1 - d_1) \right), \text{ which is the ranking formula for generalized trapezoidal fuzzy numbers [20].} \]

Corollary 3. If \( 0 \leq \lambda = \rho \leq 1 \), \( h_1 = c_1 \) and left and right membership functions of \( A \) as linear functions, then (17) reduces to

\[ R(A) = \frac{1}{4} \left( \lambda(a_1 + h_1 + c_1 + d_1) \right), \text{ which is the ranking formula for generalized triangular fuzzy numbers [20].} \]

Corollary 4. If \( 0 \leq \lambda = \rho \leq 1 \) and left and right membership functions of \( A \) as linear functions, then (17) reduces to

\[ R(A) = \frac{1}{4} \left( \lambda(a_1 + 2h_1 + d_1) \right), \text{ which is the ranking formula for normal trapezoidal fuzzy numbers [1, 20].} \]

Corollary 5. If \( 0 \leq \lambda = \rho \leq 1 \), \( h_1 = c_1 \) and left and right membership functions of \( A \) as linear functions, then (17) reduces to

\[ R(A) = \frac{1}{4} \left( \lambda(a_1 + 2h_1 + d_1) \right), \text{ which is the ranking formula for generalized triangular fuzzy numbers [1, 20].} \]

Corollary 6. If left and right membership functions of \( A \) as linear functions, then (17) reduces to

\[ R(A) = \alpha(\lambda a_1 + h_1) + \frac{1}{2} \left( \frac{(\rho - \lambda)}{2} (h_1 + a_1 + \frac{\lambda}{\rho} (h_1 - a_1)) \right) \]
Let $S$ and $S'$ be two arbitrary finite sets of exponential vague sets in which the proposed ranking function can be applied, and $A, B \in S \cap S'$, we obtain the ranking order $A \succ B$ by $\succ$ in $S$ if $A \succ B$ by $\succ$ in $S'$.

6. Results and discussion

In this section, it is proved that the proposed ranking function satisfies the reasonable properties for the ordering of fuzzy quantities. Also some examples are taken to show, shortcoming that exits in existing methods are not occurred in proposed approach.

Results of the proposed approach are compared with the existing ranking approaches.

6.1 Reasonable properties: Validation of proposed ranking approach

Wang and Kerre [33] proposed some axioms as a reasonable properties of ordering fuzzy quantities for the ranking approach $\succ$. These properties are:

**P1.** For an arbitrary subset $S$ of $F$ and $A \in S$, $A \succ A$ by $\succ$ on $S$ where, $F$ is a set of exponential vague sets.

**P2.** For an arbitrary subset $S$ of $F$ and $A, B \in S$, $A \succeq B$ and $B \succeq A$ by $\succ$ on $S$, then $A \succeq B$ by $\succ$ on $S$.

**P3.** For an arbitrary subset $S$ of $F$ and $A, B, C \in S$, $A \succ B$ and $B \succ C$ by $\succ$ on $S$, then $A \succ C$ by $\succ$ on.

**P4.** If $A \cap B = 0$ and $A$ is on the right of $B$ then $A \succ B$.

**P5.** Let $S$ and $S'$ be two arbitrary finite sets of exponential vague sets in which the proposed ranking function can be applied, and $A, B \in S \cap S'$, we obtain the ranking order $A \succ B$ by $\succ$ in $S$ if $A \succ B$ by $\succ$ in $S'$.

**P6.** If $A \succ B$ by $\succ$, then $A + C \succ B + C$ by $\succ$ when $C \neq 0$.

**P7.** Let $A, B, AC, BC$ be the elements of $S$ and $C \geq 0$. $A \succ B$ by $\succ$ if $AC \succ BC$ by $\succ$.

Now, we prove that the proposed ranking function satisfies some of the reasonable properties.

**Proposition 1.** For an arbitrary subset $S$ of $F$ and $A \in S$, $A \succ A$ by $\succ$ on $S$ where, $F$ is a set of exponential vague sets.

**Proof.** Since $A \in S \Rightarrow A, A \in S$ and $\succ$ on $S$, we have $A \succ A$ by $\succ$ on $S$.

**Proposition 2.** For an arbitrary subset $S$ of $F$ and $A, B \in S$, $A \succeq B$ and $B \succeq A$ by $\succ$ on $S$, then $A \succeq B$ by $\succ$ on $S$.

**Proof.** By proposed ranking function, $A \succeq B$ means $\succ$ and $B \succeq A$ means $\succ$, hence $\succ \Rightarrow A \succeq B$.

**Proposition 3.** For an arbitrary subset $S$ of $F$ and $A, B, C \in S$, $A \succeq B$ and $B \succeq C$ by $\succ$ on $S$, then $A \succeq C$ by $\succ$ on.

**Proof.** $A \succeq B \Rightarrow \succ \Rightarrow \succ \Rightarrow \succ \Rightarrow \succ \Rightarrow \succ \Rightarrow \succ$ Since Ranking of exponential vague sets is a crisp number i.e., the ranking function map each exponential vague set into a real line, Therefore $\succ$ in $S$ hence $A \succeq C$.
Proposition 4. If \( A \cap B = 0 \) and \( A \) is on the right of \( B \) then \( A \succ B \).

Proof. If \( A \cap B = 0 \) and \( A \) is on the right of \( B \), then \( \mathcal{R}(A) \geq \mathcal{R}(B) \) i.e., \( A \succ B \).

Proposition 5. Let \( S \) and \( S' \) be two arbitrary finite sets of exponential vague sets in which proposed ranking function can be applied, and \( A, B \in S \cap S' \), we obtain the ranking order \( A \succ B \) by \( \mathcal{R} \) in \( S \) iff \( A \succ B \) by \( \mathcal{R} \) in \( S' \).

Proof. Given that \( A \succ B \) in \( S' \) \( \Leftrightarrow \mathcal{R}(A) \geq \mathcal{R}(B) \) in \( S' \)
\( \Leftrightarrow \mathcal{R}(A) \geq \mathcal{R}(B) \) in \( S \cap S' \) (\( A, B \in S \cap S' \))
\( \Leftrightarrow \mathcal{R}(A) \geq \mathcal{R}(B) \) in \( S' \) (\( S \cap S' \subset S \)) \( \Rightarrow A \succ B \) in \( S \).

Proposition 6. If \( A \succ B \) by \( \mathcal{R} \), then \( A + C \succ B + C \)
by \( \mathcal{R} \) when \( \mathcal{R}(C) \neq 0 \) with degree of membership and non membership of \( C \) is less than or equal to the degree of membership of both, \( A \) and \( B \).

Proof. If \( A \succ B \Rightarrow \mathcal{R}(A) \geq \mathcal{R}(B) \Rightarrow \mathcal{R}(A) + \mathcal{R}(C) \geq \mathcal{R}(B) + \mathcal{R}(C) \Rightarrow \mathcal{R}(A + C) + \mathcal{R}(B + C) (\prec \) degree of membership and non membership of \( C \) is less than or equal to the degree of membership of both, \( A \) and \( B \)) \( \Rightarrow A + C \succ B + C \).

We take the examples from Section 4, and solved them with proposed ranking approach, to show that shortcomings are now removed.

Example 10. Let \( A = (0.1,0.3,0.3,0.5;0.8) \), and \( B = (0.1,0.3,0.3,0.5;1) \) be two generalized trapezoidal fuzzy numbers then \( A \Theta B = (-0.4,0,0,0,0.4;0.8) \) and \( B \Theta B = (-0.4,0,0,0,0.4;1) \)

Step 1. \( \min(0.8,1) = 0.8 \)

Step 2. \( \mathcal{R}(A \Theta B) = 0.4(0.4 - 0.8 \alpha) \)
and \( \mathcal{R}(B \Theta B) = 0.4(0.4 - 0.8 \alpha) \). Since, \( \mathcal{R}(A \Theta B) = \mathcal{R}(B \Theta B) \forall \alpha \), so \( A \Theta B \sim B \Theta B \).

Example 11. Let \( A = (0.1,0.3,0.3,0.5;1) \), and \( B = (0.3,0.5,0.5,0.7;1) \) be two generalized trapezoidal fuzzy numbers then \( A \Theta B = (-0.6,-0.2,-0.2,0.2;1) \) and \( B \Theta B = (-0.4,0,0,0,4;1) \)

Step 1. \( \min(1,1) = 1 \)

Step 2. \( \mathcal{R}(A \Theta B) = \frac{1}{2}(-0.8 \alpha \) and
\( \mathcal{R}(B \Theta B) = \frac{1}{2}(0.4 - 0.8 \alpha) \).
For a pessimistic decision maker, with \( \alpha = 0 \),
\( \mathcal{R}(A \Theta B) = 0 \) and \( \mathcal{R}(B \Theta B) = 0.2 \). Since
\( \mathcal{R}(A \Theta B) < \mathcal{R}(B \Theta B) \), so \( (A \Theta B) < (B \Theta B) \).

For optimistic decision maker, with \( \alpha = 1 \),
\( \mathcal{R}(A \Theta B) = -0.4 \) and \( \mathcal{R}(B \Theta B) = -0.2 \). Since
\( \mathcal{R}(A \Theta B) < \mathcal{R}(B \Theta B) \), so \( (A \Theta B) < (B \Theta B) \).

For moderate decision maker with \( \alpha = 0.5 \),
\( \mathcal{R}(A \Theta B) = -0.2 \) and \( \mathcal{R}(B \Theta B) = 0 \). Since
\( \mathcal{R}(A \Theta B) < \mathcal{R}(B \Theta B) \), so \( (A \Theta B) < (B \Theta B) \).

Example 12. Let \( A = (-0.5,-0.3,-0.3,-0.1;1) \), and \( B = (0.1,0.3,0.3,0.5;1) \) be two generalized trapezoidal fuzzy numbers then \( A \Theta B = (-0.6,-0.2,-0.2,0.2;1) \)
and \( B \Theta B = (-0.4,0,0,0,4;1) \)

Step 1. \( \min(1,1) = 1 \)

Step 2. \( \mathcal{R}(A \Theta B) = \frac{1}{2}(-0.8 + 2.4 \alpha \) and
\( \mathcal{R}(B \Theta B) = \frac{1}{2}(0.4 - 0.8 \alpha) \).
For a pessimistic decision maker, with \( \alpha = 0 \),
\( \mathcal{R}(A \Theta B) = -0.4 \) and \( \mathcal{R}(B \Theta B) = 0.2 \). Since
\( \mathcal{R}(A \Theta B) < \mathcal{R}(B \Theta B) \), so \( (A \Theta B) < (B \Theta B) \).

For optimistic decision maker, with \( \alpha = 1 \),
\( \mathcal{R}(A \Theta B) = -0.266 \) and \( \mathcal{R}(B \Theta B) = -0.2 \). Since
\( \mathcal{R}(A \Theta B) > \mathcal{R}(B \Theta B) \), so \( (A \Theta B) > (B \Theta B) \).
For moderate decision maker with $\alpha = 0.5$, $\mathfrak{R}(A \Theta B) = 0.2$ and $\mathfrak{R}(B \Theta B) = 0$. Since $\mathfrak{R}(A \Theta B) > \mathfrak{R}(B \Theta B)$, so $(A \Theta B) > (B \Theta B)$.

**Example 13.** Let $A = (1,1,1; w_1)$ and $B = (1,1,1; w_2)$ be two generalized trapezoidal fuzzy numbers then

**Step 1.** $\min(w_1, w_2) = w$ (say)

**Step 2.** $\mathfrak{R}(A \Theta B) = \frac{w}{2}(2\alpha + (1-\alpha)2) = w$ and $\mathfrak{R}(B \Theta B) = \frac{w}{2}(2\alpha + (1-\alpha)2) = w$.

Since, $\mathfrak{R}(A) = \mathfrak{R}(B) \ \forall \alpha$, so $A \sim B$.

**Example 14.** Let $A = (-0.4,-0.2,-0.1,0.7; w_1)$ and $B = (-0.4,-0.2,-0.1,0.7; w_2)$, be two generalized trapezoidal fuzzy numbers

**Step 1.** $\min(w_1,w_2) = w$ (say)

**Step 2.** $\mathfrak{R}(A \Theta B) = \frac{w}{2}(-0.6\alpha + (1-\alpha)0.6) = 0.3w$ and $\mathfrak{R}(B \Theta B) = \frac{w}{2}(-0.6\alpha + (1-\alpha)0.6) = 0.3w$.

Since, $\mathfrak{R}(A) = \mathfrak{R}(B) \ \forall \alpha$, so $A \sim B$.

**Example 15.** Let $A = (1.3,5;0.3,0.8)$, and $B = (4.8,9;0.4,0.9)$ be two generalized triangular vague sets then $A \Theta B = (-8,-5;0.3,0.8)$ and $B \Theta B = (-5,0.5;0.4,0.9)$.

**Step 1.** $\min(0.3,0.4) = 0.3$ and $\min(0.8,0.9) = 0.8$.

**Step 2.** $\mathfrak{R}(A \Theta B) = \frac{1}{2}(-1.037 - 3.881\alpha)$ and $\mathfrak{R}(B \Theta B) = \frac{1}{2}(2.47 - 0.94\alpha)$.

For a pessimistic decision maker, with $\alpha = 0$, $\mathfrak{R}(A \Theta B) = -1.0375$ and $\mathfrak{R}(B \Theta B) = 2.47$. Since $\mathfrak{R}(A \Theta B) < \mathfrak{R}(B \Theta B)$, so $(A \Theta B) \sim (B \Theta B)$.

For optimistic decision maker, with $\alpha = 1$, $\mathfrak{R}(A \Theta B) = -4.919$ and $\mathfrak{R}(B \Theta B) = 1.53$. Since $\mathfrak{R}(A \Theta B) < \mathfrak{R}(B \Theta B)$, so $(A \Theta B) \sim (B \Theta B)$.

**Example 16.** Let $A = (-2.0,2; \lambda_1, \rho_1)$, and $B = (-2.0,2; \lambda_2, \rho_2)$ be two triangular vague numbers then $A \Theta B = (-0.4,0.4; \lambda_3, \rho_3)$ where $\lambda_3 = \min(\lambda_1, \lambda_2)$ and $B \Theta B = (-0.4,0.4; \lambda_2, \rho_2)$.

**Step 1.** $\min(\lambda_3, \lambda_2) = \lambda$ and $\min(\rho_3, \rho_2) = \rho$

**Step 2.** Since, $\mathfrak{R}(A \Theta B) = \mathfrak{R}(B \Theta B) \ \forall \alpha$, so $A \Theta B \sim B \Theta B$.

**Example 17.** Let $A = (-8.1,4; \lambda_1, \rho_1)$, and $B = (-8.1,4; \lambda_2, \rho_2)$ be two triangular vague numbers then $A \Theta B = (-12,0.1; \lambda_3, \rho_3)$ where $\lambda_3 = \min(\lambda_1, \lambda_2)$ and $B \Theta B = (-12,0.1; \lambda_2, \rho_2)$.

**Step 1.** $\min(\lambda_3, \lambda_2) = \lambda$ and $\min(\rho_3, \rho_2) = \rho$

**Step 2.** Since, $\mathfrak{R}(A \Theta B) = \mathfrak{R}(B \Theta B) \ \forall \alpha$, so $A \Theta B \sim B \Theta B$.

**Example 18.** Let $A = (1.1; \lambda_1, \rho_1)$, and $B = (1.1; \lambda_2, \rho_2)$ be two triangular vague numbers then $A \Theta B = (0.0,0; \lambda_3, \rho_3)$ where $\lambda_3 = \min(\lambda_1, \lambda_2)$ and $B \Theta B = (0.0,0; \lambda_2, \rho_2)$.

**Step 1.** $\min(\lambda_3, \lambda_2) = \lambda$ and $\min(\rho_3, \rho_2) = \rho$

**Step 2.** Since, $\mathfrak{R}(A \Theta B) = \mathfrak{R}(B \Theta B) \ \forall \alpha$, so $A \Theta B \sim B \Theta B$.

### 6.2 Comparative study

Different sets of fuzzy sets and vague sets are taken to compare the results of proposed and existing ranking approaches.
Set 1.
Let \( A = (0.1,0.3,0.3,0.5;0.8) \) and \( B = (0.1,0.3,0.3,0.5;1) \) be two trapezoidal fuzzy sets, take \( \alpha = \frac{1}{2} \)

Step 1. \( \min(0.8,1) = 0.8 \)

Step 2. \( \mathcal{Y}(A) = 0.24 \) and \( \mathcal{Y}(B) = 0.24 \).
Since \( \mathcal{Y}(A) = \mathcal{Y}(B) \), so \( A \sim B \).

Set 2.
Let \( A = (0.1,0.3,0.3,0.5;1) \) and \( B = (0.3,0.5,0.5,0.7;1) \) be two trapezoidal fuzzy sets, take \( \alpha = \frac{1}{2} \)

Step 1. \( \min(1,1) = 1 \)

Step 2. \( \mathcal{Y}(A) = 0.3 \) and \( \mathcal{Y}(B) = 0.5 \).
Since \( \mathcal{Y}(A) < \mathcal{Y}(B) \), so \( A \prec B \).

Set 3.
Let \( A = (0.1,0.3,0.3,0.5;0.8) \) and \( B = (-0.5,-0.3,-0.3,0.1;1) \)
be two trapezoidal fuzzy sets, take \( \alpha = \frac{1}{2} \)

Step 1. \( \min(1,1) = 1 \)

Step 2. \( \mathcal{Y}(A) = 0.3 \) and \( \mathcal{Y}(B) = 0.5 \).
Since \( \mathcal{Y}(A) < \mathcal{Y}(B) \), so \( A \sim B \).

Set 4.
Let \( A = (5,7,9,11;0.4) \) and \( B = (2,4,8,9;0.2) \) be two generalized exponential trapezoidal fuzzy sets, take \( \alpha = \frac{1}{2} \)

Step 1. \( \min(0.4,0.2) = 0.2 \)

Step 2. \( \mathcal{Y}(A) = 1.6 \) and \( \mathcal{Y}(B) = 1.1 \).
Since \( \mathcal{Y}(A) > \mathcal{Y}(B) \), so \( A \succ B \).

Set 5.
Let \( A = (3,6,7,14;0.2,0.7) \) and \( A = (1,4,6,9;0.3,0.4) \) be two exponential vague fuzzy sets, take \( \alpha = \frac{1}{2} \)

Step 1. \( \min(0.2,0.3) = 0.2 \) and \( \min(0.7,0.4) = 0.4 \)

Step 2. \( \mathcal{Y}(A) = 1.52 \) and \( \mathcal{Y}(B) = 1.13 \).
Since \( \mathcal{Y}(A) > \mathcal{Y}(B) \), so \( A \succ B \).

<table>
<thead>
<tr>
<th>Methods</th>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
<th>Set 4</th>
<th>Set 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yager [36]</td>
<td>( A \sim B )</td>
<td>( A \sim B )</td>
<td>( A \sim B )</td>
<td>N.A</td>
<td>N.A</td>
</tr>
<tr>
<td>Liou and Wang [20]</td>
<td>( A \sim B )</td>
<td>( A \succ B )</td>
<td>( A \succ B )</td>
<td>N.A</td>
<td>N.A</td>
</tr>
<tr>
<td>Cheng [9]</td>
<td>( A &lt; B )</td>
<td>( A &lt; B )</td>
<td>( A \sim B )</td>
<td>N.A</td>
<td>N.A</td>
</tr>
<tr>
<td>Murakami et al. [26]</td>
<td>( A &lt; B )</td>
<td>( A &lt; B )</td>
<td>( A \succ B )</td>
<td>N.A</td>
<td>N.A</td>
</tr>
<tr>
<td>Chen and Li [7]</td>
<td>( A \sim B )</td>
<td>( A &lt; B )</td>
<td>( A \succ B )</td>
<td>( A \succ B )</td>
<td>N.A</td>
</tr>
<tr>
<td>Chen and Chen [6]</td>
<td>( A &lt; B )</td>
<td>( A &lt; B )</td>
<td>( A \succ B )</td>
<td>N.A</td>
<td>N.A</td>
</tr>
<tr>
<td>Chen and Sanguansat [8]</td>
<td>( A &lt; B )</td>
<td>( A &lt; B )</td>
<td>( A \succ B )</td>
<td>N.A</td>
<td>N.A</td>
</tr>
<tr>
<td>Li [19]</td>
<td>( A &lt; B )</td>
<td>( A &lt; B )</td>
<td>( A \succ B )</td>
<td>N.A</td>
<td>N.A</td>
</tr>
<tr>
<td>Nehi [28]</td>
<td>N.A</td>
<td>( A &lt; B )</td>
<td>( A \succ B )</td>
<td>N.A</td>
<td>N.A</td>
</tr>
<tr>
<td>Kumar et al. [15, 16, 17]</td>
<td>( A \sim B )</td>
<td>( A \sim B )</td>
<td>( A \sim B )</td>
<td>N.A</td>
<td>N.A</td>
</tr>
<tr>
<td>Proposed approach</td>
<td>( A \sim B )</td>
<td>( A \sim B )</td>
<td>( A \sim B )</td>
<td>( A \sim B )</td>
<td>( A \sim B )</td>
</tr>
</tbody>
</table>

Table 1. Comparison of proposed ranking approach with existing ranking approach where N.A denotes the not applicable.
7. Multicriteria decision-making problem based on the proposed ranking approach

Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of alternatives and let $C = \{C_1, C_2, \ldots, C_n\}$ be set of criteria. The values of an alternative on criteria $C_j(j = 1, 2, \ldots, n)$ are exponential vague sets $S_{ij}$, which indicates the degree that the alternative $A_i$ satisfies or does not satisfies the criterion $C_j$ given by decision makers or experts according to linguistic values of exponential sets for linguistic terms.

The weights of criterion $C_j(j = 1, 2, \ldots, n)$ are represented by exponential sets. The ranking weight value $w_j$ for exponential vague set is obtained by Eq. (17). The normalized weights are obtained using the following equation:

$$w_j = \frac{n}{\sum_{j=1}^{n} R(w_j)}$$  \hspace{1cm} (19)

Therefore, the weighted ranking value for an alternative $A_i(i = 1, 2, \ldots, m)$ is given by:

$$R_{w_j}(A_i) = \sum_{j=1}^{n} w_j R(S_{ij})$$  \hspace{1cm} (20)

Thus, the calculated weighted ranking value for an alternative is used to rank alternatives and then to select the best one in all the alternatives.

The above method can be summarized as follows:

(i) Calculate the ranking weight $w_j$ for criterion $C_j(j = 1, 2, \ldots, n)$ by using Eq. (17) and (18).

(ii) Calculate the weighted ranking value for alternative $A_i(i = 1, 2, \ldots, m)$ by using Eq. (17) and (19).

(iii) Rank the alternative and select the best one in accordance with weighted ranking values $R_{w_j}(A_i)$.

7.1 Illustrative example

A numerical example has been taken to show the applicability of ranking function in multicriteria decision making problem.

Suppose there is a panel with three alternative to invest the money: (i) $A_1$ is car company, (ii) $A_2$ is food company, (iii) $A_3$ is computer company.

The investment must take a decision according to the following three criteria: (i) $C_1$ is the risk analysis, (ii) $C_2$ is the growth analysis, (iii) $C_3$ is the environmental impact analysis.

The three possible alternative are to be evaluated under the above three criteria by corresponding to linguistic values of exponential sets for linguistic terms, as shown in Table 2.

<table>
<thead>
<tr>
<th>Linguistic terms</th>
<th>Linguistic values of exponential vague sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute value</td>
<td>(0.0,0.0,0.0,0.0,0.0,0.0)</td>
</tr>
<tr>
<td>Low</td>
<td>(0.0,0.1,0.2,0.3,0.1,0.2)</td>
</tr>
<tr>
<td>Fairly low</td>
<td>(0.0,0.2,0.3,0.4,0.2,0.5)</td>
</tr>
<tr>
<td>Medium</td>
<td>(0.3,0.4,0.5,0.6,0.4,0.5)</td>
</tr>
<tr>
<td>Fairly high</td>
<td>(0.5,0.6,0.7,0.8,0.5,0.7)</td>
</tr>
<tr>
<td>High</td>
<td>(0.7,0.8,0.9,1.0,0.8,0.9)</td>
</tr>
<tr>
<td>Absolutely high</td>
<td>(1.0,1.0,1.0,1.0,1.0,1.0)</td>
</tr>
</tbody>
</table>

Table 2. Linguistic values of exponential vague sets for linguistic terms.
The values of alternatives and criteria weights based on the decision makers or expert's knowledge are shown in Table 3.

Use the following steps to find the best alternative.

(i) Using Eqs. (17) and (18), we get the ranking weight value $w_j$ for criterion $C_j (j = 1, 2, 3)$,
$$w_1 = 0.383, \quad w_2 = 0.271, \quad w_3 = 0.345.$$

(ii) Using Eqs. (17) and (19), we get the weighted ranking value for alternative
$$A_i (i = 1, 2, 3), \quad R_{w_1}(A_1) = 0.901, \quad R_{w_2}(A_2) = 0.778, \quad R_{w_3}(A_3) = 0.937.$$

(iii) Rank the alternative as follows:
$$A_1 < A_2 < A_3$$

Thus according to above results the most desirable alternative is $A_3$.

8. Conclusions

In this paper, we present a new approach for the ranking of exponential vague sets. New representation and arithmetic operations between two exponential vague sets has been introduced. Some shortcomings of existing ranking approaches [8, 19] are pointed out. The proposed method consider not only the rank but also the decision maker optimistic attitude also with the help of some comparative examples it is shown that proposed ranking approach is more intuitive than the existing ranking approaches. Also the proposed ranking function satisfies the reasonable properties for the ordering of fuzzy quantity. Further the proposed ranking approach can effectively rank of various types of fuzzy sets and vague sets (normal, generalized, triangular, trapezoidal and exponential), which is another advantage of proposed ranking approach over the other exiting ranking approaches. For practical use, proposed ranking approach is applied to decision making problem.

<table>
<thead>
<tr>
<th></th>
<th>$k$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1</td>
<td>(0.1,0.2,0.3,0.4,0.2,0.3)</td>
<td>(0.2,0.3,0.5,0.7,0.3,0.6)</td>
<td>(0.3,0.4,0,6,0.7,0.8,0.9)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(0.3,0.5,0.6,0.8,0.5,0.7)</td>
<td>(0.2,0.3,0.5,0.7,0.3,0.6)</td>
<td>(0.3,0.4,0.6,0.7,0.8,0.9)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(0.3,0.5,0.6,0.8,0.5,0.7)</td>
<td>(0.2,0.3,0.5,0.7,0.3,0.6)</td>
<td>(0.3,0.5,0.6,0.8,0.5,0.7)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>1</td>
<td>(0.3,0.4,0.6,0.7,0.8,0.9)</td>
<td>(0.1,0.2,0.3,0.4,0.2,0.3)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(0.3,0.4,0.6,0.7,0.8,0.9)</td>
<td>(0.2,0.3,0.5,0.7,0.3,0.6)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(0.1,0.2,0.3,0.4,0.2,0.3)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>1</td>
<td>(0.2,0.3,0.5,0.7,0.3,0.6)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
<td>(0.2,0.3,0.5,0.7,0.3,0.6)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
<td>(0.3,0.5,0.6,0.8,0.5,0.7)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(0.3,0.5,0.6,0.8,0.5,0.7)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
</tr>
<tr>
<td>weight</td>
<td>1</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
<td>(0.2,0.3,0.5,0.7,0.3,0.6)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(0.3,0.4,0.6,0.7,0.8,0.9)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
<td>(0.1,0.4,0.6,0.7,0.5,0.6)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(0.3,0.5,0.6,0.8,0.5,0.7)</td>
<td>(0.3,0.4,0.6,0.7,0.8,0.9)</td>
<td>(0.3,0.4,0.6,0.7,0.8,0.9)</td>
</tr>
</tbody>
</table>

Table 3. Linguistic values of exponential vague sets for linguistic terms.
Acknowledgements

The author is very grateful to the Editor-in-Chief, and the anonymous referees, for their constructive comments and suggestions that led to an improved version of this paper. The author was supported by the Education for Competitiveness Operational Programme project “Encourage the creation of excellent research teams and intersectoral mobility at Palacky University in Olomouc” reg. no. CZ.1.07/2.3.00/30.0004, which is co-financed by the European Social Fund and the state budget of the Czech Republic.

References


[21] A. Lu and W. Ng, “Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better?” Lecture Notes in Computer Science, 3716, 401-416, 2005.


