Solving the Partial Differential Problems Using Maple

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ABSTRACT
This paper considers the partial differential problem of two types of multivariable functions and uses mathematical software Maple for verification. The infinite series forms of any order partial derivatives of these two types of multivariable functions can be obtained using binomial series and differentiation term by term theorem, which greatly reduce the difficulty of calculating their higher order partial derivative values. On the other hand, four examples are used to demonstrate the calculations.

Keywords: Partial derivatives, infinite series forms, binomial series, differentiation term by term theorem, Maple.

1. Introduction

In calculus and engineering mathematics, the evaluation and numerical calculation of the partial derivatives of multivariable functions are important. The Laplace equation, the wave equation, and other important physical equations involve the partial derivatives. The evaluation of the m -th order partial derivative value of a multivariable function at some point, generally, requires two procedures: the determination of the m -th order partial derivative of the function, and the substitution of the point into the m -th order partial derivative. These two procedures become increasingly complex calculations for increasing order of partial derivative, thus manual calculations become difficult. The present study considers the partial differential problem of the following two types of n -variables functions

$$f(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} x_i^{\beta_i} \cdot \left(a + b \prod_{i=1}^{n} x_i^{\lambda_i} \right)^r \quad (1)$$

$$g(x_1, x_2, \cdots, x_n) = \exp\left(\sum_{i=1}^{n} \beta_i x_i \right) \cdot \left(a + b \exp\left(\sum_{i=1}^{n} \lambda_i x_i \right) \right)^r \quad (2)$$

where n is a positive integer, $a, b, r, \beta_i, \lambda_i$ are real numbers for all $i = 1,.., n$, $a, b \neq 0$, and a^r, b^r exist. We can obtain the infinite series forms of any order partial derivatives of these two types of multivariable functions using binomial series and differentiation term by term theorem; these are the major results of this study (i.e., Theorems 1 and 2), which greatly reduce the difficulty of calculate their higher order partial derivative values. The study of partial differential problems can refer to [1-24]. The methods adopted in [1-5] are different from the methods used in this paper, and [6-24] studied the evaluation of the partial derivatives of different types of multivariable functions using differentiation term by term theorem and complex power series method. [25] considered two differential equations whose independent variables involve the partial derivatives. [26] discussed the distance functions whose expressions contain the partial derivatives, and [27] found the solutions of some type of partial differential equation. In this article, some examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

2. Main Results

Some notations used in this paper are introduced below.

2.1 Notations

2.1.1

$\prod_{i=1}^{n} c_i = c_1 \times c_2 \times \cdots \times c_n$, where n is a positive integer, c_i are real numbers for all $i = 1,..,n$.

2.1.2

Suppose that t is any real number, and m is any positive integer. Define

$(t)_m = t(t-1)\cdots(t-m+1)$, and $(t)_0 = 1$.

2.1.3

Suppose that n is a positive integer, j_i are non-negative integers for all $i = 1,..,n$. For the n-variables function $f(x_1, x_2, \cdots, x_n)$, its j_i times partial derivative with respect to x_i for all $i = 1,..,n$, forms a $j_1 + j_2 + \cdots + j_n$-th order partial derivative, denoted as

$$\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n)$$

The followings are two important theorems used in this study.

2.2 Binomial series

$(1+u)^r = \sum_{k=0}^{\infty} \frac{(r)_k}{k!} u^k$, where u, r are real numbers, and $|u| < 1$.

2.3 Differentiation term by term theorem ([28, p230]).

For all non-negative integers k, if the functions $g_k : (a,b) \to R$ satisfy the following three conditions: (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on the open interval (a,b), and (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b), then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b). Moreover, its derivative $\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

The following is the first major result in this study, we determine the infinite series forms of any order partial derivatives of the n-variables function (1).

2.4 Theorem 1

Suppose that n is a positive integer, $a, b, r, \lambda_i, \beta_i$ are real numbers for all $i = 1,..,n$, $a, b \neq 0$, and a^r, b^r exist. If the n-variables function

$$f(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} x_i^{\beta_i} \cdot \left(a + b \prod_{i=1}^{n} x_i^{\lambda_i}\right)^r$$

satisfies that $x_i^{\beta_i}, x_i^{\lambda_i}, x_i^{\lambda_i r}$ exist, $x_i \neq 0$ for all $i = 1,..,n$, and $\prod_{i=1}^{n} x_i^{\lambda_i} \neq \pm \frac{a}{b}$.

Case A. If $\left|\prod_{i=1}^{n} x_i^{\lambda_i}\right| < \left|\frac{a}{b}\right|$, then the $j_1 + j_2 + \cdots + j_n$-th order partial derivative of $f(x_1, x_2, \cdots, x_n)$

$$\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n)$$

$$= a^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!} \left(\frac{b}{a}\right)^k \prod_{i=1}^{n} (\lambda_i k + \beta_i)_{j_i} \cdot \prod_{i=1}^{n} x_i^{\lambda_i k + \beta_i - j_i} \tag{3}$$

Case B. If $\left|\prod_{i=1}^{n} x_i^{\lambda_i}\right| > \left|\frac{a}{b}\right|$ $\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n)$

$$= b^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{a}{b}\right)^k \prod_{i=1}^{n}(-\lambda_i k+\beta_i+\lambda_i r)_{j_i} \cdot \prod_{i=1}^{n} x_i^{-\lambda_i k+\beta_i+\lambda_i r-j_i} \tag{4}$$

Proof Case A. If $\left|\prod_{i=1}^{n} x_i^{\lambda_i}\right| < \left|\frac{a}{b}\right|$

Because

$$f(x_1, x_2, \cdots, x_n)$$
$$= \prod_{i=1}^{n} x_i^{\beta_i} \cdot \left(a + b\prod_{i=1}^{n} x_i^{\lambda_i}\right)^r$$
$$= \prod_{i=1}^{n} x_i^{\beta_i} \cdot a^r\left(1 + \frac{b}{a}\prod_{i=1}^{n} x_i^{\lambda_i}\right)^r$$
$$= \prod_{i=1}^{n} x_i^{\beta_i} \cdot a^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k \prod_{i=1}^{n} x_i^{\lambda_i k}$$

(Because $\left|\prod_{i=1}^{n} x_i^{\lambda_i}\right| < \left|\frac{a}{b}\right|$, we can use binomial series)

$$= a^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k \prod_{i=1}^{n} x_i^{\lambda_i k+\beta_i} \tag{5}$$

By differentiation term by term theorem, differentiating j_i-times with respect to x_i ($i = 1,.., n$) on both sides of Equation (5), we have: the $j_1 + j_2 + \cdots + j_n$-th order partial derivative of $f(x_1, x_2, \cdots, x_n)$, $\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n)$

$$= a^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k \prod_{i=1}^{n}(\lambda_i k+\beta_i)_{j_i} \cdot \prod_{i=1}^{n} x_i^{\lambda_i k+\beta_i-j_i}$$

Case B. If $\left|\prod_{i=1}^{n} x_i^{\lambda_i}\right| > \left|\frac{a}{b}\right|$

Because

$$f(x_1, x_2, \cdots, x_n)$$
$$= \prod_{i=1}^{n} x_i^{\beta_i} \cdot \left(a + b\prod_{i=1}^{n} x_i^{\lambda_i}\right)^r$$
$$= \prod_{i=1}^{n} x_i^{\beta_i} \cdot b^r \prod_{i=1}^{n} x_i^{\lambda_i r}\left(1 + \frac{a}{b}\prod_{i=1}^{n} x_i^{-\lambda_i}\right)^r$$
$$= \prod_{i=1}^{n} x_i^{\beta_i+\lambda_i r} \cdot b^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{a}{b}\right)^k \prod_{i=1}^{n} x_i^{-\lambda_i k}$$
$$= b^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{a}{b}\right)^k \prod_{i=1}^{n} x_i^{-\lambda_i k+\beta_i+\lambda_i r} \tag{6}$$

Using differentiation term by term theorem, differentiating j_i-times with respect to x_i ($i = 1,.., n$) on both sides of Equation (6), we obtain:

$$\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n)$$
$$= b^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{a}{b}\right)^k \prod_{i=1}^{n}(-\lambda_i k+\beta_i+\lambda_i r)_{j_i} \cdot \prod_{i=1}^{n} x_i^{-\lambda_i k+\beta_i+\lambda_i r-j_i}$$

■

2.5 Remark 1

If $\left|\prod_{i=1}^{n} x_i^{\lambda_i}\right| < \left|\frac{a}{b}\right|$, using ratio test ([28, p193]) yields:

$$a^r \cdot \sum_{k=0}^{\infty} \frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k \prod_{i=1}^{n}(\lambda_i k+\beta_i)_{j_i} \cdot \prod_{i=1}^{n} x_i^{\lambda_i k+\beta_i-j_i}$$

is uniformly convergent. Thus, we can use differentiation term by term theorem to prove Equation (3) holds. The same reason that we can employ differentiation term by term theorem to confirm Equation (4) holds.

The following is the second major result in this study, we obtain the infinite series forms of any order partial derivatives of the n -variables function (2).

2.6 Theorem 2

If the assumptions are the same as Theorem 1. Suppose that the n -variables function

$$g(x_1, x_2, \cdots, x_n) = \exp\left(\sum_{i=1}^{n} \beta_i x_i\right) \cdot \left(a + b\exp\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right)^r$$

satisfies that $\left(a + b\exp\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right)^r$ exists, and

$$\exp\left(\sum_{i=1}^{n}\lambda_i x_i\right) \neq \left|\frac{a}{b}\right|$$

Case A. If $\exp\left(\sum_{i=1}^{n}\lambda_i x_i\right) < \left|\frac{a}{b}\right|$, then the $j_1+j_2+\cdots+j_n$-th order partial derivative of $g(x_1,x_2,\cdots,x_n)$,

$$\frac{\partial^{j_1+j_2+\cdots+j_n} g}{\partial x_n^{j_n}\cdots\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,\cdots,x_n) = a^r\cdot\sum_{k=0}^{\infty}\frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k\prod_{i=1}^{n}(\lambda_i k+\beta_i)^{j_i}\cdot\exp\left(\sum_{i=1}^{n}(\lambda_i k+\beta_i)x_i\right) \tag{7}$$

Case B. If $\exp\left(\sum_{i=1}^{n}\lambda_i x_i\right) > \left|\frac{a}{b}\right|$, then

$$\frac{\partial^{j_1+j_2+\cdots+j_n} g}{\partial x_n^{j_n}\cdots\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,\cdots,x_n) = b^r\cdot\sum_{k=0}^{\infty}\frac{(r)_k}{k!}\left(\frac{a}{b}\right)^k\prod_{i=1}^{n}(-\lambda_i k+\beta_i+\lambda_i r)^{j_i}\times$$

$$\exp\left(\sum_{i=1}^{n}(-\lambda_i k+\beta_i+\lambda_i r)x_i\right) \tag{8}$$

Proof Case A. If $\exp\left(\sum_{i=1}^{n}\lambda_i x_i\right) < \left|\frac{a}{b}\right|$

Because

$$\begin{aligned} g(x_1,x_2,\cdots,x_n) &= \exp\left(\sum_{i=1}^{n}\beta_i x_i\right)\cdot\left(a+b\exp\left(\sum_{i=1}^{n}\lambda_i x_i\right)\right)^r \\ &= \exp\left(\sum_{i=1}^{n}\beta_i x_i\right)\cdot a^r\left(1+\frac{b}{a}\exp\left(\sum_{i=1}^{n}\lambda_i x_i\right)\right)^r \\ &= \exp\left(\sum_{i=1}^{n}\beta_i x_i\right)\cdot a^r\cdot\sum_{k=0}^{\infty}\frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k\exp\left(k\sum_{i=1}^{n}\lambda_i x_i\right) \\ &= a^r\cdot\sum_{k=0}^{\infty}\frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k\exp\left(\sum_{i=1}^{n}(\lambda_i k+\beta_i)x_i\right) \end{aligned} \tag{9}$$

By differentiation term by term theorem, differentiating j_i-times with respect to x_i ($i=1,..,n$) on both sides of Equation (9), we obtain: the $j_1+j_2+\cdots+j_n$-th order partial derivative of $g(x_1,x_2,\cdots,x_n)$,

$$\frac{\partial^{j_1+j_2+\cdots+j_n} g}{\partial x_n^{j_n}\cdots\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,\cdots,x_n) = a^r\cdot\sum_{k=0}^{\infty}\frac{(r)_k}{k!}\left(\frac{b}{a}\right)^k\prod_{i=1}^{n}(\lambda_i k+\beta_i)^{j_i}\cdot\exp\left(\sum_{i=1}^{n}(\lambda_i k+\beta_i)x_i\right)$$

Case B. $\exp\left(\sum_{i=1}^{n}\lambda_i x_i\right) > \left|\frac{a}{b}\right|$

Because

$$\begin{aligned} g(x_1,x_2,\cdots,x_n) &= \exp\left(\sum_{i=1}^{n}\beta_i x_i\right)\cdot\left(a+b\exp\left(\sum_{i=1}^{n}\lambda_i x_i\right)\right)^r \\ &= \exp\left(\sum_{i=1}^{n}\beta_i x_i\right)\cdot b^r\exp\left(r\sum_{i=1}^{n}\lambda_i x_i\right)\left(1+\frac{a}{b}\exp\left(-\sum_{i=1}^{n}\lambda_i x_i\right)\right)^r \\ &= \exp\left(\sum_{i=1}^{n}\beta_i x_i\right)\cdot b^r\exp\left(r\sum_{i=1}^{n}\lambda_i x_i\right)\left(1+\frac{a}{b}\exp\left(-\sum_{i=1}^{n}\lambda_i x_i\right)\right)^r \\ &= b^r\exp\left(\sum_{i=1}^{n}(\beta_i+r\lambda_i)x_i\right)\cdot\sum_{k=0}^{\infty}\frac{(r)_k}{k!}\left(\frac{a}{b}\right)^k\exp\left(-k\sum_{i=1}^{n}\lambda_i x_i\right) \\ &= b^r\cdot\sum_{k=0}^{\infty}\frac{(r)_k}{k!}\left(\frac{a}{b}\right)^k\exp\left(\sum_{i=1}^{n}(-\lambda_i k+\beta_i+\lambda_i r)x_i\right) \end{aligned} \tag{10}$$

Using differentiation term by term theorem, differentiating j_i-times with respect to x_i ($i=1,..,n$) on both sides of Equation (10), we obtain:

$$\frac{\partial^{j_1+j_2+\cdots+j_n} g}{\partial x_n^{j_n}\cdots\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,\cdots,x_n) = b^r\cdot\sum_{k=0}^{\infty}\frac{(r)_k}{k!}\left(\frac{a}{b}\right)^k\prod_{i=1}^{n}(-\lambda_i k+\beta_i+\lambda_i r)^{j_i}\times$$

$$\exp\left(\sum_{i=1}^{n}(-\lambda_i k+\beta_i+\lambda_i r)x_i\right)$$

■

2.7 Remark 2

The same reason as that in Remark 1, we can use differentiation term by term theorem to prove Equations (7) and (8) hold.

3. Examples

For the partial differential problem of the multivariable functions in this study, four examples are proposed. Theorems 1 and 2 are used to obtain the infinite series forms of any order partial derivatives of these functions, and to evaluate some of their higher order partial derivative values. Additionally, Maple is used to calculate the approximations of these higher order partial derivative values to verify the manual calculations.

3.1 Example 1

Suppose that the domain of the two-variables function

$$f_1(x_1,x_2)=x_1^{7/3}x_2^{5/2}\cdot(9+2x_1^3x_2^4)^{11/5} \quad (11)$$

is $\left\{(x_1,x_2)\in R^2 \,\middle|\, x_1\neq 0, x_2>0, x_1^3x_2^4\neq\pm\frac{9}{2}\right\}$

3.1.1

If $\left|x_1^3x_2^4\right|<\frac{9}{2}$. Using Case A of Theorem 1 yields: any j_1+j_2-th order partial derivative of $f_1(x_1,x_2)$,

$$\frac{\partial^{j_1+j_2}f_1}{\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2)=9^{\frac{11}{5}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{11}{5}\right)_k}{k!}\left(\frac{2}{9}\right)^k\left(3k+\frac{7}{3}\right)_{j_1}\left(4k+\frac{5}{2}\right)_{j_2}\times x_1^{3k+\frac{7}{3}-j_1}\cdot x_2^{4k+\frac{5}{2}-j_2} \quad (12)$$

Therefore, the 13-th order partial derivative value of $f_1(x_1,x_2)$ at $\left(\frac{3}{2},\frac{4}{5}\right)$,

$$\frac{\partial^{13}f_1}{\partial x_2^7\partial x_1^6}\left(\frac{3}{2},\frac{4}{5}\right)=9^{\frac{11}{5}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{11}{5}\right)_k}{k!}\left(\frac{2}{9}\right)^k\left(3k+\frac{7}{3}\right)_6\left(4k+\frac{5}{2}\right)_7\times\left(\frac{3}{2}\right)^{3k-\frac{11}{3}}\cdot\left(\frac{4}{5}\right)^{4k-\frac{9}{2}} \quad (13)$$

Next, we use Maple to verify the correctness of Equation (13).

```
>f1:=(x1,x2)->x1^(7/3)*x2^(5/2)*(9+2*x1^3*x2^4)
^(11/5);

>evalf(D[1$6,2$7](f1)(3/2,4/5),22);
```

$$7.123326797821678044703\cdot 10^{11}$$

```
>evalf(9^(11/5)*sum(product(11/5-j,j=0..(k-1))/k!*
(2/9)^k*product(3*k+7/3-p,p=0..5)*product(4*k+5/2-
q,q=0..6)*(3/2)^(3*k-11/3)*(4/5)^(4*k-9/2),k=0..
infinity),22);
```

$$7.123326797821678044705\cdot 10^{11}$$

3.1.2

If $\left|x_1^3x_2^4\right|>\frac{9}{2}$. By Case B of Theorem 1, the j_1+j_2-th order partial derivative of $f_1(x_1,x_2)$,

$$\frac{\partial^{j_1+j_2}f_1}{\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2)=2^{\frac{11}{5}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{11}{5}\right)_k}{k!}\left(\frac{9}{2}\right)^k\left(-3k+\frac{134}{15}\right)_{j_1}\left(-4k+\frac{113}{10}\right)_{j_2}\times x_1^{-3k+\frac{134}{15}-j_1}\cdot x_2^{-4k+\frac{113}{10}-j_2} \quad (14)$$

Thus, the 8-th order partial derivative value of $f_1(x_1,x_2)$ at $(2,3)$,

$$\frac{\partial^8 f_1}{\partial x_2^4 \partial x_1^4}(2,3)$$

$$=2^{\frac{11}{5}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{11}{5}\right)_k}{k!}\left(\frac{9}{2}\right)^k\left(-3k+\frac{134}{15}\right)_4\left(-4k+\frac{113}{10}\right)_4\times$$

$$2^{-3k+\frac{74}{15}}\cdot 3^{-4k+\frac{73}{10}} \tag{15}$$

```
>evalf(D[1$4,2$4](f1)(2,3),18);
```

$1.11843022977422795\cdot 10^{13}$

```
>evalf(2^(11/5)*sum(product(11/5-j,j=0..(k-1))/k!*(9/2)^k*product(-3*k+134/15-p,p=0..3)*product(-4*k+113/10-q,q=0..3)*2^(-3*k+74/15)*3^(-4*k+73/10),k=0..infinity),18);
```

$1.11843022977422796\cdot 10^{13}$

3.2 Example 2

Assume that the domain of the three-variables function

$$f_2(x_1,x_2,x_3)=\frac{x_1^{11/6}x_2^{13/4}x_3^{8/5}}{(11-3x_1^2x_2^5x_3^4)^{2/3}} \tag{16}$$

is $\left\{(x_1,x_2,x_3)\in R^3 \middle| x_1>0,x_2>0,x_3\neq 0,x_1^2x_2^5x_3^4\neq\pm\frac{11}{3}\right\}$

3.2.1

If $\left|x_1^2x_2^5x_3^4\right|<\frac{11}{3}$. Using Case A of Theorem 1, we obtain: any $j_1+j_2+j_3$-th order partial derivative of $f_2(x_1,x_2,x_3)$,

$$\frac{\partial^{j_1+j_2+j_3} f_2}{\partial x_3^{j_3}\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,x_3)$$

$$=11^{\frac{-2}{3}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{-2}{3}\right)_k}{k!}\left(\frac{-3}{11}\right)^k\left(2k+\frac{11}{6}\right)_{j_1}\left(5k+\frac{13}{4}\right)_{j_2}\left(4k+\frac{8}{5}\right)_{j_3}\times$$

$$x_1^{2k+\frac{11}{6}-j_1}\cdot x_2^{5k+\frac{13}{4}-j_2}\cdot x_3^{4k+\frac{8}{5}-j_3} \tag{17}$$

Thus, the 15-th order partial derivative value of $f_2(x_1,x_2,x_3)$ at $\left(\frac{1}{6},\frac{2}{3},\frac{3}{4}\right)$,

$$\frac{\partial^{15} f_2}{\partial x_3^5\partial x_2^6\partial x_1^4}\left(\frac{1}{6},\frac{2}{3},\frac{3}{4}\right)$$

$$=11^{\frac{-2}{3}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{-2}{3}\right)_k}{k!}\left(\frac{-3}{11}\right)^k\left(2k+\frac{11}{6}\right)_4\left(5k+\frac{13}{4}\right)_6\left(4k+\frac{8}{5}\right)_5\times$$

$$\left(\frac{1}{6}\right)^{2k-\frac{13}{6}}\cdot\left(\frac{2}{3}\right)^{5k-\frac{11}{4}}\cdot\left(\frac{3}{4}\right)^{4k-\frac{17}{5}} \tag{18}$$

```
>f2:=(x1,x2,x3)->x1^(11/6)*x2^(13/4)*x3^(8/5)/((11-3*x1^2*x2^5*x3^4)^2)^(1/3);
```

```
>evalf(D[1$4,2$6,3$5](f2)(1/6,2/3,3/4),14);
```

$5.3261668131349\cdot 10^{7}$

```
>evalf(11^(-2/3)*sum(product(-2/3-j,j=0..(k-1))/k!*(-3/11)^k*product(2*k+11/6-p,p=0..3)*product(5*k+13/4-q,q=0..5)*product(4*k+8/5-s,s=0..4)*(1/6)^(2*k-13/6)*(2/3)^(5*k-11/4)*(3/4)^(4*k-17/5),k=0..infinity),14);
```

$5.3261668131350\cdot 10^{7}$

3.2.2

If $\left|x_1^2x_2^5x_3^4\right|>\frac{11}{3}$. Using Case B of Theorem 1 yields: any $j_1+j_2+j_3$-th order partial derivative of $f_2(x_1,x_2,x_3)$,

$$\frac{\partial^{j_1+j_2+j_3} f_2}{\partial x_3^{j_3}\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,x_3)$$

$$=(-3)^{\frac{-2}{3}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{-2}{3}\right)_k}{k!}\left(\frac{-11}{3}\right)^k\left(-2k+\frac{1}{2}\right)_{j_1}\left(-5k-\frac{1}{12}\right)_{j_2}\left(-4k-\frac{16}{15}\right)_{j_3}\times$$

$$x_1^{-2k+\frac{1}{2}-j_1}\cdot x_2^{-5k-\frac{1}{12}-j_2}\cdot x_3^{-4k-\frac{16}{15}-j_3} \quad (19)$$

Hence, the 11-th order partial derivative value of $f_2(x_1,x_2,x_3)$ at $(4,2,5)$,

$$\frac{\partial^{11} f_2}{\partial x_3^6 \partial x_2^2 \partial x_1^3}(4,2,5)$$
$$=3^{\frac{-2}{3}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{-2}{3}\right)_k}{k!}\left(\frac{-11}{3}\right)^k\left(-2k+\frac{1}{2}\right)_3\left(-5k-\frac{1}{12}\right)_2\left(-4k-\frac{16}{15}\right)_6\times$$
$$4^{-2k-\frac{5}{2}}\cdot 2^{-5k-\frac{25}{12}}\cdot 5^{-4k-\frac{106}{15}} \quad (20)$$

>evalf(D[1$3,2$2,3$6](f2)(4,2,5),14);

-0.000019108587954897

>evalf(3^(-2/3)*sum(product(-2/3-j,j=0..(k-1))/k!*(-11/3)^k*product(-2*k+1/2-p,p=0..2)*product(-5*k-1/12-q,q=0..1)*product(-4*k-16/15-s,s=0..5)*4^(-2*k-5/2)*2^(-5*k-25/12)*5^(-4*k-106/15),k=0..infinity),14);

-0.000019108587954897

3.3 Example 3

If the domain of the two-variables function

$$g_1(x_1,x_2)=\exp(2x_1+3x_2)\cdot[7+9\exp(5x_1+8x_2)]^{\frac{13}{3}} \quad (21)$$

is $\left\{(x_1,x_2)\in R^2 \,\middle|\, \exp(5x_1+8x_2)\neq\frac{7}{9}\right\}$

3.3.1

If $\exp(5x_1+8x_2)<\frac{7}{9}$. By Case A of Theorem 2, we obtain: any j_1+j_2-th order partial derivative of $g_1(x_1,x_2)$,

$$\frac{\partial^{j_1+j_2} g_1}{\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2)$$
$$=7^{\frac{13}{3}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{13}{3}\right)_k}{k!}\left(\frac{9}{7}\right)^k(5k+2)^{j_1}(8k+3)^{j_2}\times$$
$$\exp[(5k+2)x_1+(8k+3)x_2] \quad (22)$$

Therefore, the 12-th order partial derivative value of $g_1(x_1,x_2)$ at $\left(-\frac{1}{4},-\frac{2}{5}\right)$,

$$\frac{\partial^{12} g_1}{\partial x_2^7 \partial x_1^5}\left(-\frac{1}{4},-\frac{2}{5}\right)$$
$$=7^{\frac{13}{3}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{13}{3}\right)_k}{k!}\left(\frac{9}{7}\right)^k(5k+2)^5(8k+3)^7\cdot\exp\left(\frac{-89}{20}k-\frac{17}{10}\right) \quad (23)$$

>g1:=(x1,x2)->exp(2*x1+3*x2)*(7+9*exp(5*x1+8*x2))^(13/3);

>evalf(D[1$5,2$7](g1)(-1/4,-2/5),22);

$5.855345231176575279854\cdot 10^{14}$

>evalf(7^(13/3)*sum(product(13/3-j,j=0..(k-1))/k!*(9/7)^k*(5*k+2)^5*(8*k+3)^7*exp(-89/20*k-17/10),k=0..infinity),22);

$5.855345231176575279879\cdot 10^{14}$

3.3.2

If $\exp(5x_1+8x_2)>\frac{7}{9}$. Using Case B of Theorem 2 yields: any j_1+j_2-th order partial derivative of $g_1(x_1,x_2)$,

$$\frac{\partial^{j_1+j_2} g_1}{\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2)$$
$$=9^{\frac{13}{3}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{13}{3}\right)_k}{k!}\left(\frac{7}{9}\right)^k\left(-5k+\frac{71}{3}\right)^{j_1}\left(-8k+\frac{113}{3}\right)^{j_2}\times$$
$$\exp\left[\left(-5k+\frac{71}{3}\right)x_1+\left(-8k+\frac{113}{3}\right)x_2\right] \quad (24)$$

Therefore, we can determine the 17-th order partial derivative value of $g_1(x_1, x_2)$ at $\left(\frac{7}{2}, \frac{1}{6}\right)$,

$$\frac{\partial^{17} g_1}{\partial x_2{}^9 \partial x_1{}^8}\left(\frac{7}{2}, \frac{1}{6}\right)$$

$$= 9^{\frac{13}{3}} \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{13}{3}\right)_k}{k!}\left(\frac{7}{9}\right)^k \left(-5k + \frac{71}{3}\right)^8 \left(-8k + \frac{113}{3}\right)^9 \times$$

$$\exp\left(\frac{-113}{6}k + \frac{802}{9}\right) \quad (25)$$

Maple was used to verify the correctness of Equation (25):

>evalf(D[1$8,2$9](g1)(7/2,1/6),22);

$1.028551772875461091101 \cdot 10^{68}$

>evalf(9^(13/3)*sum(product(13/3-j,j=0..(k-1))/k!*(7/9)^k*(-5*k+71/3)^8*(-8*k+113/3)^9*exp(-113/6*k+802/9),k=0..infinity),22);

$1.028551772875461091120 \cdot 10^{68}$

3.4 Example 4

Suppose that the domain of the three-variables function

$$g_2(x_1, x_2, x_3) = \frac{\exp(6x_1 + 11x_2 - 8x_3)}{[14 - 5\exp(7x_1 - 4x_2 + 2x_3)]^{8/9}} \quad (26)$$

is $\left\{(x_1, x_2, x_3) \in R^3 \middle| \exp(7x_1 - 4x_2 + 2x_3) \neq \frac{14}{5}\right\}$

3.4.1

If $\exp(7x_1 - 4x_2 + 2x_3) < \frac{14}{5}$. By Case A of Theorem 2, we obtain: any $j_1 + j_2 + j_3$ -th order partial derivative of $g_2(x_1, x_2, x_3)$,

$$\frac{\partial^{j_1+j_2+j_3} g_2}{\partial x_3{}^{j_3} \partial x_2{}^{j_2} \partial x_1{}^{j_1}}(x_1, x_2, x_3)$$

$$= 14^{\frac{-8}{9}} \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{-8}{9}\right)_k}{k!}\left(\frac{-5}{14}\right)^k (7k+6)^{j_1}(-4k+11)^{j_2}(2k-8)^{j_3} \times$$

$$\exp[(7k+6)x_1 + (-4k+11)x_2 + (2k-8)x_3] \quad (27)$$

Thus, the 14-th order partial derivative value of $g_2(x_1, x_2, x_3)$ at $\left(-\frac{1}{7}, \frac{2}{3}, -\frac{1}{2}\right)$,

$$\frac{\partial^{14} g_2}{\partial x_3{}^3 \partial x_2{}^5 \partial x_1{}^6}\left(-\frac{1}{7}, \frac{2}{3}, -\frac{1}{2}\right)$$

$$= 14^{\frac{-8}{9}} \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{-8}{9}\right)_k}{k!}\left(\frac{-5}{14}\right)^k (7k+6)^6(-4k+11)^5(2k-8)^3 \times$$

$$\exp\left(\frac{-14}{3}k + \frac{220}{21}\right) \quad (28)$$

>g2:=(x1,x2,x3)->exp(6*x1+11*x2-8*x3)/((14-5*exp(7*x1-4*x2+2*x3))^8)^(1/9);

>evalf(D[1$6,2$5,3$3](g2)(-1/7,2/3,-1/2),22);

$-1.324276036594542259720 \cdot 10^{16}$

>evalf(14^(-8/9)*sum(product(-8/9-j,j=0..(k-1))/k!*(-5/14)^k*(7*k+6)^6*(-4*k+11)^5*(2*k-8)^3*exp(-14/3*k+220/21),k=0..infinity),22);

$-1.324276036594542259721 \cdot 10^{16}$

3.4.2

If $\exp(7x_1 - 4x_2 + 2x_3) > \frac{14}{5}$. Using Case B of Theorem 2 yields: any $j_1 + j_2 + j_3$ -th order partial derivative of $g_2(x_1, x_2, x_3)$,

$$\frac{\partial^{j_1+j_2+j_3} g_2}{\partial x_3{}^{j_3} \partial x_2{}^{j_2} \partial x_1{}^{j_1}}(x_1, x_2, x_3)$$

$$= 5^{\frac{-8}{9}} \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{-8}{9}\right)_k}{k!}\left(\frac{-14}{5}\right)^k \left(-7k - \frac{2}{9}\right)^{j_1} \left(4k + \frac{131}{9}\right)^{j_2} \left(-2k - \frac{88}{9}\right)^{j_3} \times$$

$$\exp\left[\left(-7k-\frac{2}{9}\right)x_1+\left(4k+\frac{131}{9}\right)x_2+\left(-2k-\frac{88}{9}\right)x_3\right] \qquad (29)$$

Hence, the 16-th order partial derivative value of $g_2(x_1,x_2,x_3)$ at $\left(\frac{1}{7},\frac{1}{4},1\right)$,

$$\frac{\partial^{16} g_2}{\partial x_3{}^4 \partial x_2{}^7 \partial x_1{}^5}\left(\frac{1}{7},\frac{1}{4},1\right)$$

$$=5^{\frac{-8}{9}}\cdot\sum_{k=0}^{\infty}\frac{\left(\frac{-8}{9}\right)_k}{k!}\left(\frac{-14}{5}\right)^k\left(-7k-\frac{2}{9}\right)^5\left(4k+\frac{131}{9}\right)^7\left(-2k-\frac{88}{9}\right)^4\times$$

$$\exp\left(-2k-\frac{1555}{252}\right) \qquad (30)$$

>evalf(D[1$5,2$7,3$4](g2)(1/7,1/4,1),40);

$$-7.227518225794619524748080\cdot 10^{20}$$

>evalf(5^(-8/9)*sum(product(-8/9-j,j=0..(k-1))/k!*(-14/5)^k*(-7*k-2/9)^5*(4*k+131/9)^7*(-2k-88/9)^4*exp(-2*k-1555/252),k=0..infinity),22);

$$-7.227518225794619524748\cdot 10^{20}$$

4. Conclusion

This article proposed two methods (i.e., the binomial series and the differentiation term by term theorem) to solve the partial differential problem of some multivariable functions. The two methods can be applied to evaluate any order partial derivatives of general multivariable functions. Further studies on related applications will be conducted in the future. Moreover, other calculus and engineering mathematics problems will be considered and solved using Maple.

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